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ON A VARIATIONAL FORMULATION OF A CLASS OF THERMAL ENTRANCE PROBLEMS

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Abstract—It is shown that a class of constant property, unsteady, laminar flow thermal entrance problems, in which the velocity profile can be considered to be fully developed, may be formulated as a variational problem in the Laplace transformed domain. In turn the variational formulation allows the application of the powerful Ritz–Galerkin technique to generate approximate solutions. The technique proposed is illustrated by an application to the Graetz problem between semi-infinite parallel plates and to the entrance problem in which the wall temperature varies sinusoidally.

NOMENCLATURE

a_k,	kth approximating polynomial coefficient;
c_p,	specific heat;
d,	half width of channel between parallel plates;
d_e,	equivalent diameter (=4 cross-sectional area/wetted perimeter);
g_k,	kth approximating polynomial function;
G,	mass velocity ($=\rho U$)
h_l,	local heat-transfer coefficient;
k,	thermal conductivity;
Nu,	local Nusselt number;
Nu_{avg},	average Nusselt number;
p, q,	Laplace transform parameters;
Pr,	Prandtl number;
q'',	heat flux rate;
Re,	Reynolds number;
T,	temperature;
u^*,	axial velocity component;
U,	average axial velocity;
x, y, z,	cartesian coordinates, z being axial to the flow;
α,	thermal diffusivity;
ε,	sinusoidal wall temperature amplitude;
γ,	cross-section of the cylindrical region Γ;
Γ,	semi-infinite cylindrical region;
η, ξ, ζ, τ,	non-dimensional cartesian coordinates and time;
θ,	non-dimensional temperature;
λ,	wall temperature distribution wave length
Λ,	closed curve in the plane $z = 0$ demarcating the region γ;
μ,	viscosity;
ρ,	density;
σ,	variational parameter;
ϕ,	non-dimensional axial location, also the Graetz number;
Φ,	non-dimensional source function;
ω,	non-dimensional wave number.

INTRODUCTION

In designing heat transfer equipment involving laminar forced convective heat transfer inside tubes and ducts, it is customary to artificially divide the tube or duct into two regions, the entrance region and the, so called, fully developed region. In general, the entrance problems are considerably more difficult to handle than those in which the thermal and velocity fields are assumed to be fully established

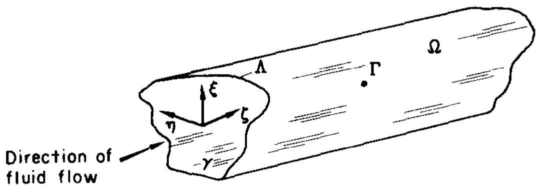

FIG. 1. Semi-infinite cylindrical region Γ.

(including the case of slug flow), that is, unchanging in the direction of the flow. The purpose of the present paper is to establish that a class of thermal entrance problems can be formulated in the Laplace-transformed domain as variational problems, which in turn allows the application of the powerful method of Ritz and Galerkin [1] to generate approximate solutions. The technique suggested herein is very useful especially when dealing either with complex geometries and/or boundary conditions not conveniently handled by standard techniques. Arpaci and Vest [2] had applied this technique to problems of diffusion and Sparrow and Siegel [7] had previously devised a variational principle for thermal entrance problems, but had restricted themselves to the steady case. The present paper presents a more generalized variational formulation applicable to a wider class of problems.

THE VARIATIONAL PROBLEM

Consider the class of constant property thermal entrance problems defined by the following formulation consisting of the unsteady energy equation:

$$\frac{\partial\theta}{\partial\tau} + u^*(\eta, \xi)\frac{\partial\theta}{\partial\zeta} = \nabla^2_{\eta\xi}\,\theta + \Phi(\eta, \xi, \zeta, \tau) \text{ in } \Gamma \qquad (1)$$

with the boundary and initial conditions on the temperature θ given as

$$\theta(\eta, \xi, \zeta, 0) = 0, \qquad \theta(\eta, \xi, 0, \tau) = 0,$$
$$\theta = \beta(\Lambda, \zeta, \tau) \text{ on } \Omega. \qquad (2)$$

That is wall temperature specified. With reference to Fig. 1, equation (1) is defined to hold in the semi-infinite cylindrical region Γ contained within the surface Ω formed by a generator parallel to the ζ axis and whose base traverses the curve Λ in the plane $\zeta = 0$. The planar region defined by Λ is γ. The symbol $\nabla^2_{\eta\xi}$ is used to define the operator $(\partial^2/\partial\eta^2 + \partial^2/\partial\xi^2)$, thus we have neglected axial conduction. The symbol "Φ" defines internal heat sources (for example as a result of a radioactive solution or due to the passage of a current through an electrolyte, etc.). The fully developed axial velocity component is denoted by $u^*(\eta, \xi)$.

The assumption of fully developed velocity profile is generally valid for fluids of high Prandtl numbers (such as oils) or in those cases where the heated section is preceded by an unheated hydrodynamic development section.

Taking the double Laplace transform of equations (1) and (2) with respect to τ and ζ, defined by

$$\left.\begin{aligned} \tilde{f}(\eta, \xi, \zeta, q) &= \int_0^\infty e^{-q\tau} f(\eta, \xi, \zeta, \tau)\,\mathrm{d}\tau \\ \tilde{\tilde{f}}(\eta, \xi, p, q) &= \int_0^\infty e^{-p\zeta} \tilde{f}(\eta, \xi, \zeta, q)\,\mathrm{d}\zeta \end{aligned}\right\} \qquad (3)$$

where the Laplace transformation parameters are p and q, we get on rearranging

$$\left.\begin{aligned} &\nabla^2_{\eta\xi}\,\tilde{\tilde{\theta}} - (u^* p + q)\,\tilde{\tilde{\theta}} + \Phi = \tilde{0} \text{ defined in } \gamma \\ &\tilde{\tilde{\theta}} = \tilde{\tilde{\beta}} \text{ on } \Lambda. \end{aligned}\right. \qquad (4)$$

The symbol "$\approx$" over θ and Φ are meant to indicate the Laplace transformed functions.

We now show that equation (4) is the Euler equation of a variational problem. But before proceeding with that, it is to be noted that, for the sake of simplicity, the discussion here has been restricted to the boundary condition of prescribed wall temperature. However the two other boundary conditions commonly used in the analysis of thermal entrance problems, that of prescribed heat flux at the wall ($\partial\theta/\partial n = \beta^*$, where $\partial\theta/\partial n$ denotes the normal derivative on the wall) or a combination of the above two ($\partial\theta/\partial n + b\theta = \beta^{**}$) may be treated in a like manner with the proper choice of the functional "I" (See Appendix A).

Consider now the integral

$$I(\tilde{\tilde{\theta}}) = \int\int_{\gamma} \left[\left(\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta}\right)^2 + \left(\frac{\partial\tilde{\tilde{\theta}}}{\partial\xi}\right)^2 + (u^* p + q)\tilde{\tilde{\theta}}^2 - 2\tilde{\tilde{\Phi}}\tilde{\tilde{\theta}} \right] d\eta\, d\zeta. \quad (5)$$

We seek the functions $\tilde{\theta}$ which subject to the condition $\tilde{\tilde{\theta}} = \tilde{\tilde{\beta}}$ on Λ lead to a minimum value of $I(\tilde{\tilde{\theta}})$. Suppose that $\tilde{\tilde{\theta}}(\eta, \xi, p, q)$ is such a function. Thus any other function $\tilde{\tilde{\theta}} + \sigma\psi$, such that $|\sigma|$ is sufficiently small and $\psi(\eta, \xi, p, q)$ is continuous together with its first partial derivatives with respect to η, ξ in γ and such that $\psi = 0$ on Λ, would lead to

$$I(\tilde{\tilde{\theta}} + \sigma\psi) > I(\tilde{\tilde{\theta}}) \quad (6)$$

or that the variation δI of the integral, given by $[dI(\tilde{\tilde{\theta}} + \sigma\psi)/d\sigma]_{\sigma=0}$, is equal to zero.

$$\delta I = [dI(\tilde{\tilde{\theta}} + \sigma\psi)/d\sigma]_{\sigma=0} = 0. \quad (7)$$

That is

$$\delta I = \int\int_{\gamma} \left[2\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta}\frac{\partial\psi}{\partial\eta} + 2\frac{\partial\tilde{\tilde{\theta}}}{\partial\xi}\frac{\partial\psi}{\partial\xi} + 2(u^* p + q)\tilde{\tilde{\theta}}\psi - 2\tilde{\tilde{\Phi}}\psi \right] d\eta\, d\xi = 0. \quad (8)$$

Using Green's formula [3], the first two terms may be transformed to obtain

$$\delta I = 2\int_{\Lambda} \left(\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta}\psi\, d\xi - \frac{\partial\tilde{\tilde{\theta}}}{\partial\xi}\psi\, d\eta \right) + 2\int\int_{\gamma} \left[-\left(\frac{\partial^2\tilde{\tilde{\theta}}}{\partial\eta^2} + \frac{\partial^2\tilde{\tilde{\theta}}}{\partial\xi^2} \right) + (u^* p + q)\tilde{\tilde{\theta}} - \tilde{\tilde{\Phi}} \right] \psi\, d\eta\, d\xi = 0. \quad (9)$$

The first integral is identically zero since $\psi \equiv 0$ on Λ. Hence $\delta I = 0$ implies that the second integral is equal to zero, for any arbitrary function ψ which is non-zero somewhere in γ. The only way that is possible is if the integrand itself is zero, and since ψ is arbitrary, equation (4) is obtained.

Thus we see that the Laplace transformed form of equation (1), with appropriate boundary conditions, is identical to the Euler equation of the integral I, equation (5). With a variational principle established, approximate solutions to system (4) may be generated using the Ritz–Galerkin technique.

THE RITZ–GALERKIN APPROXIMATION

We shall here only present the formal procedure used in the Ritz–Galerkin technique of approximation. The details of the mathematical problems of convergence, uniqueness of solution, etc., that arise can be found in [1].

Suppose $\tilde{\tilde{\theta}}_n$ represents the nth order approximation, then in accordance with Ritz–Galerkin procedure

$$\tilde{\tilde{\theta}}_n = \tilde{\tilde{\theta}}_0 + \sum_{k=1}^{n} a_k g_k \quad (10)$$

where $\tilde{\tilde{\theta}}_0$ satisfies the nonhomogeneous boundary condition, $\tilde{\tilde{\theta}}_0 = \tilde{\tilde{\beta}}$ on Λ and where the set of trial functions $\{g_k\}$ is relatively complete* in the region γ and such that the boundary conditions, $\tilde{\tilde{\theta}} = 0$ on Λ, are satisfied by the remainder of equation (10). Then the coefficients a_k may be determined as the solution to the system of "n" simultaneous equations

$$\int\int_{\gamma} [\nabla^2_{\eta\xi}\tilde{\tilde{\theta}}_n - (u^* p + q)\tilde{\tilde{\theta}}_n + \tilde{\tilde{\Phi}}]\, g_k\, d\eta\, d\xi = 0 \qquad k = 1, 2, \ldots, n \quad (11)$$

which results from the substitution of $\tilde{\tilde{\theta}}_n$ for $\tilde{\tilde{\theta}}$ in (5) and applying the conditions leading to the minimum of the integral I, i.e.

$$\frac{\partial I}{\partial a_k} = 0 \qquad \text{for } k = 1, 2, \ldots, n. \quad (12)$$

The relative completeness condition will generally be satisfied in the class of problems

* See [1] pages 258–262 for further details.

posed by choosing the "g_k" as either trigonometric functions or polynomials which satisfy the boundary conditions. The relative completeness of the set $\{g_k\}$ is a sufficient condition to guarantee that the approximate solution $\tilde{\tilde{\theta}}_n$ will yield successively closer approximations to the minimum of the integral I as $n \to \infty$ and $\lim_{n\to\infty} \tilde{\tilde{\theta}}_n = \tilde{\tilde{\theta}}$, where $\tilde{\tilde{\theta}}$ is the exact solution of equation (4). The approximate solution to equations (1) and (2) are obtained by the inversion of $\tilde{\tilde{\theta}}_n$.

In order to show the applicability of the technique and the method of solution to be used, we illustrate the above by obtaining approximate solutions to the Graetz problem between infinite parallel plates and the entrance problem with sinusoidally varying wall temperature.

RITZ–GALERKIN APPROXIMATION TO CLASSICAL GRAETZ PROBLEM

Consider first the classical Graetz problem between semi-infinite parallel plates (see Fig. 2).

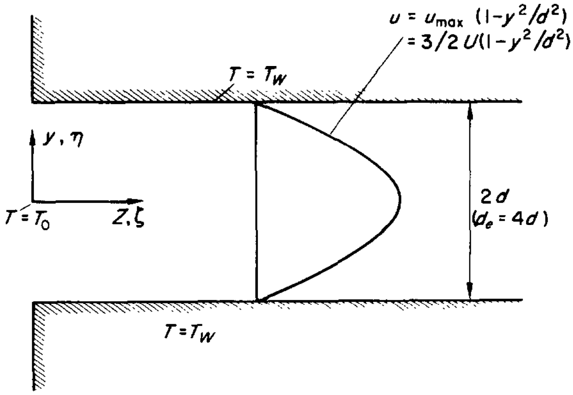

FIG. 2. Semi-infinite flat rectangular duct.

In non-dimensional terms the problem may be posed as the determination of the function θ governed by the equation:

$$(1 - \eta^2)\frac{\partial \theta}{\partial \zeta} = \frac{\partial^2 \theta}{\partial \eta^2} \tag{13}$$

where

$$\eta = \frac{y}{d}, \qquad \theta = \frac{T - T_0}{T_w - T_0}, \qquad \zeta = \frac{z}{d\,Re\,Pr}$$

$$Re = \tfrac{4}{3}\,Ud\,\rho/\mu \qquad Pr = \mu c_p/k.$$

The symbols d, T_0, T_w, U signify respectively the half width of the channel, uniform entrance temperature, uniform wall temperature and the average flow velocity ($=\frac{2}{3}u_{max}$). Re and Pr are a Reynolds number and the Prandtl number of the fluid whose properties are constant. The viscosity, thermal conductivity, specific heat density of the fluid are denoted respectively by μ, k, c_p and ρ.

The appropriate initial and boundary conditions are

$$\theta(0, \eta) = 0, \qquad \theta(\zeta, 1) = 1, \qquad (\partial\theta/\partial\eta)_{\eta=0} = 0. \tag{14}$$

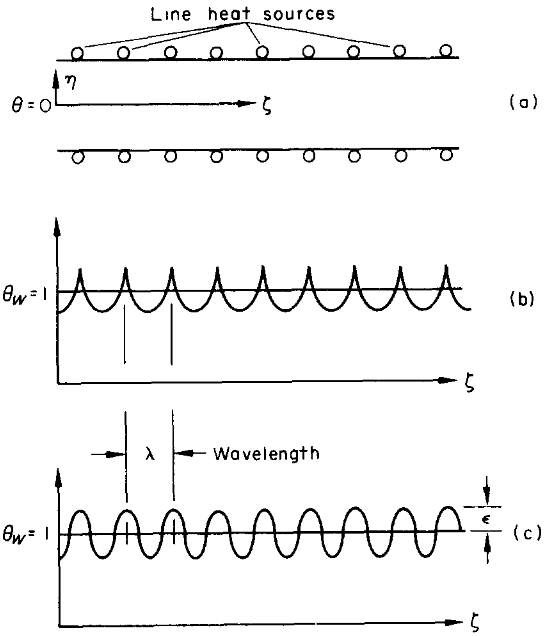

FIG. 3.

Taking the Laplace transform of equation (13) with respect to ζ we get the ordinary differential equation

$$\frac{d^2\bar{\theta}}{d\eta^2} - p\bar{\theta}(1 - \eta^2) = 0 \tag{15}$$

subject to the conditions

$$\bar{\theta}(p, 1) = 1/p \qquad \text{and} \qquad (d\bar{\theta}/dn)_{\eta=0} = 0. \tag{16}$$

For the set $\{g_k\}$ choose the linearly independent set of polynomials satisfying the homogeneous boundary conditions $\tilde{\theta}(p, 1) = 0$ and $(\mathrm{d}\tilde{\theta}/\mathrm{d}\eta)_{\eta=0} = 0$.

$$g_k = (1 - \eta^2)\,\eta^{2(k-1)} \quad k = 1, 2, \ldots, n \tag{17}$$

and thus seek the approximate solution of the form

$$\tilde{\theta}_n^* = 1/p + \sum_{k=1}^{n} a_k(1 - \eta^2)\,\eta^{2(k-1)} \tag{18}$$

Note that $\tilde{\theta}_n^*$ satisfies the boundary conditions in (16). For the sake of brevity we reproduce here only the key steps in computation and those only for the second order approximation ($n = 2$). Thus a_1 and a_2 will be obtained by solving simultaneously the system of equations [obtained by applying equation (11)].

$$\int_0^1 \left[\frac{\mathrm{d}^2\tilde{\theta}_n^*}{\mathrm{d}\eta^2} - p\tilde{\theta}_n^*(1 - \eta^2) - (1 - \eta^2)\right] (1 - \eta^2)\,\eta^{2(k-1)}\,\mathrm{d}\eta = 0 \qquad k = 1, 2. \tag{19}$$

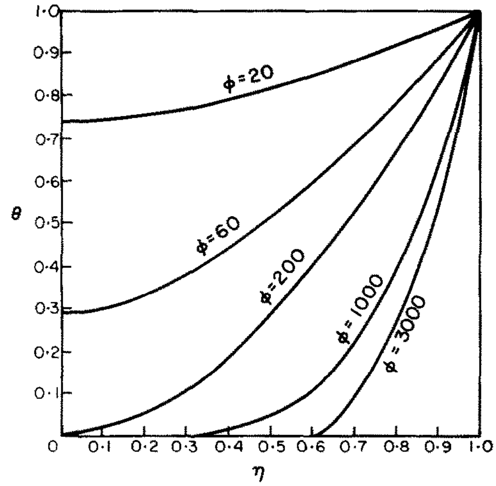

FIG. 4. Non-dimensional temperature plots.

On carrying out the indicated integration, obtain

$$\left(\frac{1}{3} + \frac{4}{35}p\right) a_1 + \left(\frac{1}{15} + \frac{4}{315}p\right) a_2 = -\frac{2}{15} \tag{20}$$

$$\left(1 + \frac{4}{21}p\right) a_1 + \left(\frac{11}{7} + \frac{4}{77}p\right) a_2 = -\frac{2}{7} \tag{21}$$

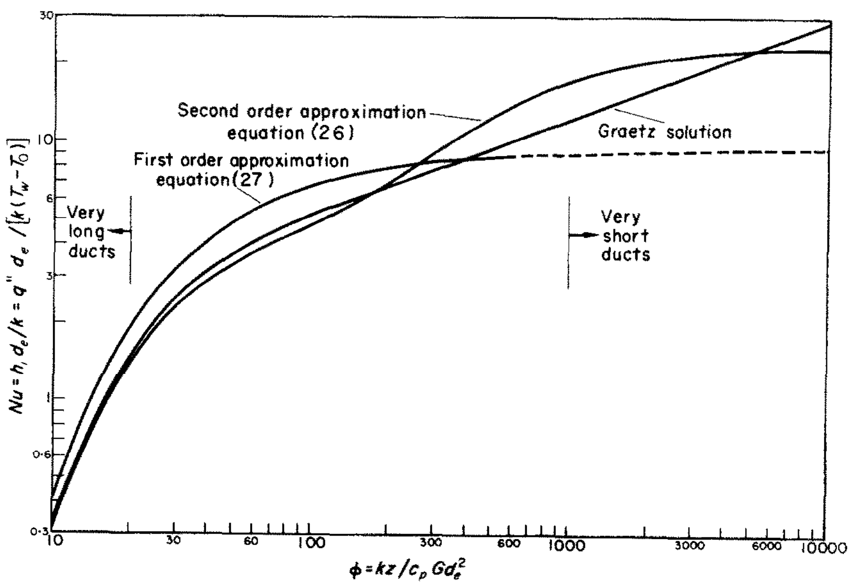

FIG. 5. Local Nusselt numbers.

from which we obtain the following approximation for $\tilde{\theta}$

$$\tilde{\theta} = \frac{1}{p} + (1 - \eta^2)\,[-(0{\cdot}1906 + 0{\cdot}0033\,p) + (0{\cdot}0382 - 0{\cdot}00726\,p)]/(0{\cdot}457 + 0{\cdot}174\,p + 0{\cdot}0035\,p^2). \quad (22)$$

Inverting equation (22) with respect to p we get as the approximation for θ:

$$\theta = 1 + (1 - \eta^2)\,e^{-24{\cdot}7\zeta}\,[\sinh(21{\cdot}9\,\zeta)\,\{2{\cdot}83\,\eta^2 - 1{\cdot}415\} - \cosh(21{\cdot}9\,\zeta)\,\{2{\cdot}064\,\eta^2 + 0{\cdot}936\}]. \quad (23)$$

Equation (23) is plotted in Fig. 4, using the independent variable ϕ sometimes referred to as the Graetz number

$$\phi = \frac{c_p\,G\,d_e^2}{kz} = \frac{32}{3\zeta}. \quad (24)$$

Using the definition of local Nusselt number

$$Nu = h_l d_e/k = q'' d_e/\{(T_w - T_0)\,k\} = 4\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \quad (25)$$

where d_e is the equivalent diameter of the duct $(=4d)$, we obtain from (23)

$$Nu = 8\,e^{-24{\cdot}7\zeta}\,[3\cosh(21{\cdot}9\,\zeta) - 1{\cdot}42\sinh(21{\cdot}9\,\zeta)]. \quad (26)$$

Equation (26) is compared with the classical Graetz solution obtained from [4, 5] in Fig. 5. Also shown is the first order solution obtained by assuming $\tilde{\theta}_1^* = 1/p + a_1(1 - \eta^2)\,\eta^2$, which yields

$$Nu = \frac{28}{3}\exp\left(-\frac{35\,\zeta}{12}\right). \quad (27)$$

ENTRANCE PROBLEM WITH SINUSOIDAL WALL TEMPERATURE DISTRIBUTION

Consider now the second example which we pose as follows:
Suppose in a given experimental investigation we are forced to simulate the uniform wall temperature using a series of line heat sources or discrete heating strips (see Fig. 3a). As would be expected, the wall temperature distribution will be non-uniform and spatially cyclic (as in Fig. 3b). We wish to determine the effect of this non-uniform temperature distribution on, for example, the measured Nusselt numbers. To do this we formulate the problem mathematically by using a sinusoidal approximation to the wall temperature distribution (as in Fig. 3c). Hence we are required to solve equation (13) subject to the conditions

$$\theta(0, \eta) = 0, \qquad \theta(\zeta, 1) = 1 + \varepsilon\sin(\omega\zeta),$$
$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 0 \quad (28)$$

where ε is the dimensionless amplitude of the sinusoidal distribution about the mean wall temperature $(\theta_w = 1)$ and ω is the dimensionless wave number defined as

$$\omega = \frac{2\pi d\,Re\,Pr}{\lambda} \quad (29)$$

in which λ is the wavelength of the assumed sinusoid (or physically the spacing between the line heat sources).

An exact solution to this problem can in fact be obtained from the constant wall temperature solution through the transformation presented by Sellers *et al.* [4] but the calculations involved are considerably more involved and have not been carried out to date.

The second order approximation for Nusselt numbers (local) obtained by solving equations (13) and (28) using the proposed method is obtained to be

$$Nu = 8\left[e^{-24{\cdot}7\zeta}\left\{\cosh(21{\cdot}9\zeta)\left(3 + \frac{43{\cdot}2\,\omega\varepsilon}{\omega^2 + 479{\cdot}6}\right) + \sinh(21{\cdot}9\zeta)\left(\frac{65{\cdot}7\,\omega\varepsilon}{\omega^2 + 479{\cdot}6} - 1{\cdot}42\right)\right\} + \frac{\omega\varepsilon}{\omega^2 + 479{\cdot}6}\{3\omega\sin(\omega\zeta) - 43{\cdot}2\cos(\omega\zeta)\}\right]. \quad (30)$$

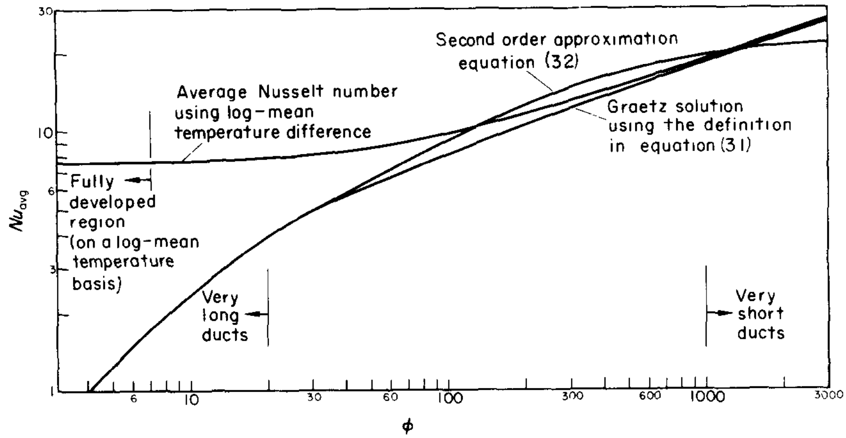

FIG. 6. Average Nusselt numbers.

DISCUSSION OF NUMERICAL RESULTS

Consider first the Graetz problem. From Figs. 4 and 5 it is obvious that the suggested technique does indeed yield an approximate solution to the Graetz problem over a given region and that the higher order approximation yields increasingly closer fit. We note in particular that the requirement of fitting Nusselt numbers or the first derivative at the wall is a good deal more severe condition than merely approximating the temperature field in magnitude.

Since we have need to define a thermal entrance length, we must first of all define the fully developed condition. In the strictest sense, the fully developed condition for the constant wall temperature boundary condition ($\partial T/\partial z = 0$) is only reached when the fluid achieves a uniform temperature the same as the wall and the process is mathematically asymptotic in nature. However, if the log-mean temperature difference (see [5]) is used to define the average heat-transfer coefficient, the average Nusselt number in the limit $\zeta \to \infty$ ($\phi \to 0$) approaches a value of 7·6 (Fig. 6). On this basis we may arbitrarily define the entrance region as being the region in which the limiting value of the log-mean Nusselt number is approached within 1·0 per cent. In our problem this condition yields the entrance region as corresponding to $\phi \gtrsim 7$ ($\zeta \lesssim 1{\cdot}524$). In the discussion to follow, we will therefore define the entrance region to be $\phi \gtrsim 7$.

With the above definition in mind, the 2nd order approximation (for local Nusselt numbers) is generally accurate to within 15 per cent in all but the initial 3 per cent of the entrance region, that is, in the range $\phi < 300$. The deviations from the Graetz solution for $\phi > 400$ are considerable, but the error changes sign at $\phi \approx 5000$. It is to be expected that the third order approximation will yield a uniformly better approximation (although the algebra becomes increasingly more formidable). However, it is characteristic of the method suggested that the solution for very small ζ (large ϕ) and τ will be in considerable error. Despite this shortcoming, it is noted that the solutions for both the function sought and its first derivative can be accurately assessed over a major portion of the duct. Indeed the integrated average heat-transfer coefficient shows a much better agreement over a larger range of ϕ. The average coefficient for the 2nd order solution defined by

$$Nu_{avg} = \frac{1}{\zeta}\int_0^{\zeta} Nu(x)\,dx \tag{31}$$

$$= \phi(0{\cdot}247 - 0{\cdot}212\,e^{-30/\phi} - 0{\cdot}0355\,e^{-497/\phi}) \tag{32}$$

is compared with the average coefficient obtained from Sparrow [6] in Fig. 6. The maximum error in the range $\phi < 3000$ (more than 99 per cent of the entrance region) is on the order of 15 per cent and less than 5 per cent in the range $\phi < 60$ (long ducts).

We turn now to the problem of the sinusoidal wall temperature distribution. Using the definition in equation (31) we obtain for the average Nusselt number

$$Nu_{\text{avg}} = \frac{4}{\zeta}\left[0{\cdot}659 + 38{\cdot}42\frac{\omega\varepsilon}{\omega^2 + 479{\cdot}6} - \left(0{\cdot}564 + 38{\cdot}9\frac{\omega\varepsilon}{\omega^2 + 479{\cdot}6}\right)e^{-2{\cdot}8\zeta} - \left(0{\cdot}0946 - 0{\cdot}483\frac{\omega\varepsilon}{\omega^2 + 479{\cdot}6}\right)e^{-46{\cdot}6\zeta} + \frac{\varepsilon}{\omega^2 + 479{\cdot}6}\{6\omega(1 - \cos(\omega\zeta)) - 86{\cdot}4\sin(\omega\zeta)\}\right]. \quad (33)$$

Equation (33) is compared with equation (32) for a range of wave numbers in Fig. (7) for $\varepsilon = 0{\cdot}5$. Equation (33) is again plotted in Fig. 8 for a fixed value of $\phi = 10{\cdot}0$ ($\zeta = 1{\cdot}066$), nearly fully developed condition, for several ε values.

As is to be expected, maximum deviation from the uniform wall temperature case decreases as the wave number $(\omega) \to \infty$ and increases as ε increases from 0. Indeed, except for the region in the immediate entrance region ($\phi > 1000$), there is very little difference between the solutions for $\varepsilon = 0$ and $\omega > 1000$. Thus we have a "rule of the thumb" criterion for designing the experimental apparatus for the problem posed, to minimize errors (if $\varepsilon > 0$) ω must be > 1000. This implies that for a given λ/d ratio, the Reynolds number (using equivalent diameter and average velocity) must exceed the value given by

$$\frac{\rho U d_e}{\mu} > \frac{8}{3}\left(\frac{1000\,(\lambda/d)}{2\pi\, Pr}\right). \quad (34)$$

This in turn implies that the worst problem will occur in tests involving gases ($Pr \sim 1$). As a case in point, for air, with $Pr \sim 0{\cdot}72$ and $\lambda/d = 1$. the Reynolds number must be greater than or equal to roughly 600, and even so the error in the

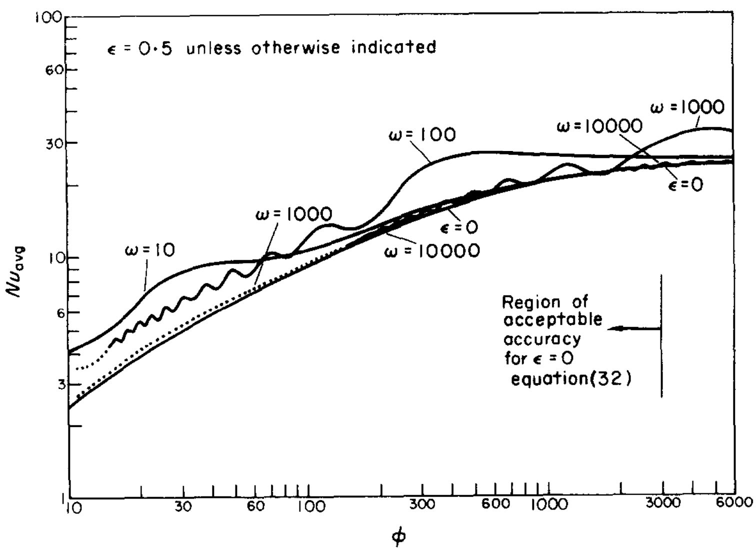

FIG. 7. Average numbers with wave numbers as a parameter.

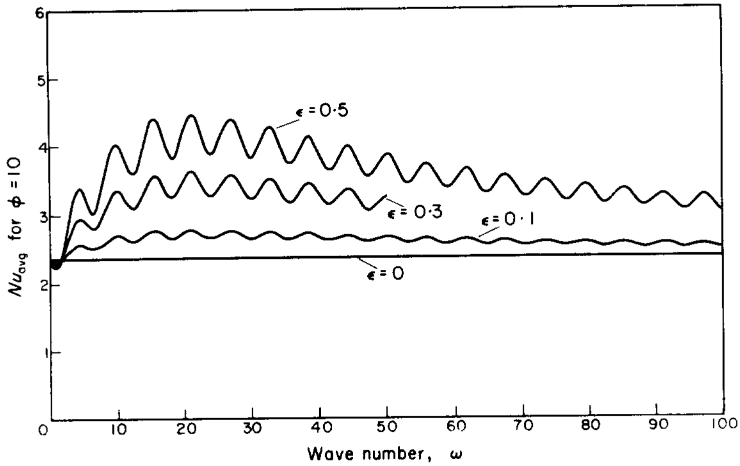

FIG. 8. Average Nusselt number vs. wave number for $\phi = 10$.

Nusselt number at $\phi = 10$ and $\varepsilon = 0{\cdot}1$ will amount to $+6$ per cent. This is an error inherent in the system even prior to measurement errors.

We note that the formulation of the original differential equations (1) and (13) fail to account for conduction in the z or ζ direction. Thus the solutions so obtained cannot be applied to liquid metals (very low Prandtl numbers).

In conclusion, we have shown that the transformed thermal entrance problem can be cast in a variational form, which in turn allows the application of the Ritz–Galerkin approximation technique. The technique yields acceptable solutions which show an improved accuracy as the order of approximation is increased. The technique is especially valuable in the case of those thermal entrance problems (defined by equations (1) and (2)) which are not conveniently handled by the classical techniques. Similarly, if θ in equation (1) is interpreted as a concentration, the above technique will apply equally well to a class of problems of mass transfer.

Finally, it should be pointed out that the method suggested herein is equivalent to the application of Galerkin method using the trial functions $\theta = \sum A_k(\tau, \zeta)\, g_k(\eta, \xi)$. For further related discussion the reader is referred to the discussion by Goodman of Erdogan's paper [8].

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APPENDIX A

Varational Formulation for the Mixed Boundary Condition

We consider here the case of the mixed condition. We wish to show that the equation

$$\nabla^2_{\eta\zeta}\tilde{\tilde{\theta}} - (u^* p + q)\tilde{\tilde{\theta}} + \tilde{\tilde{\Phi}} = 0 \tag{A.1}$$

is an Euler equation to a variational problem subject to the boundary condition

$$\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta} + b\tilde{\tilde{\theta}} = \tilde{\tilde{\beta}}^{**} \tag{A.2}$$

Consider the functional

$$I(\tilde{\tilde{\theta}}) = \int_\gamma\!\!\int\left[\left(\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta}\right)^2 + \left(\frac{\partial\tilde{\tilde{\theta}}}{\partial\zeta}\right)^2 + (u^* p + q)\tilde{\tilde{\theta}}^2 - 2\tilde{\tilde{\Phi}}\tilde{\tilde{\theta}}\right]\mathrm{d}\eta\,\mathrm{d}\zeta + \int_A (b\tilde{\tilde{\theta}}^2 - \tilde{\tilde{2}}\beta^{**}\tilde{\tilde{\theta}})\,\mathrm{d}s. \tag{A.3}$$

Then if $\tilde{\tilde{\theta}}$ minimizes I and $(\tilde{\tilde{\theta}} + \sigma\Psi)$ is a function such that $|\sigma|$ is sufficiently small and Ψ is continuous together with its first partial derivatives in γ, we obtain as the variation of I after using the Green's formula as in equation (8):

$$\delta I = 2\int_A\left\{\frac{\partial\tilde{\tilde{\theta}}}{\partial\eta}\frac{\mathrm{d}\zeta}{\mathrm{d}s} - \frac{\partial\tilde{\tilde{\theta}}}{\partial\zeta}\frac{\mathrm{d}n}{\mathrm{d}s} + b\tilde{\tilde{\theta}} - \tilde{\tilde{\beta}}^{**}\right\}\Psi\,\mathrm{d}s$$

$$+2\int_\gamma\!\!\int\left[-\left(\frac{\partial^2\tilde{\tilde{\theta}}}{\partial\eta^2} + \frac{\partial^2\tilde{\tilde{\theta}}}{\partial\zeta}\right) + (u^* p + q)\tilde{\tilde{\theta}} - \tilde{\tilde{\Phi}}\right]\Psi\,\mathrm{d}\eta\,\mathrm{d}\zeta = 0. \tag{A.4}$$

Considering the line integral above we note it may be written as

$$\int_A\left\{\frac{\partial\tilde{\tilde{\theta}}}{\partial n} + b\tilde{\tilde{\theta}} - \tilde{\tilde{\beta}}^{**}\right\}\Psi\,\mathrm{d}s. \tag{A.5}$$

Now in view of the boundary condition (A.2), the integral in (A.5) is identically zero. Then, in view of the arbitrary nature of ψ we obtain (A.1) as the Euler equation of (A.3) subject to the condition (A.2).

Finally note that $b = 0$ describes the prescribed heat flux boundary condition. The formal procedure of the Ritz–Galerkin method hereafter remains the same as section III above, only the boundary condition to be satisfied by equation (10) is (A.2).

FORMULATION VARIATIONNELLE D'UNE CLASSE DE PROBLÈMES D'ENTRÉE THERMIQUE

Résumé—On montre qu'une classe de problèmes d'écoulement d'entrée thermique laminaire, instationnaire et à propriétés constantes, dans laquelle le profil de vitesse peut être considéré comme étant entièrement développé, peut être formulée comme un problème variationnel dans le domaine des transformées de Laplace. En outre, la formulation variationnelle permet l'application de la puissante technique de Ritz–Galerkin pour obtenir des solutions approchées. La technique proposée est illustrée par une application au problème de Graetz entre des plaques parallèles semi-infinies et au problème d'entrée dans lequel la température pariétale varie sinusoïdalement.

EINE VARIATIONSFORMULIERUNG FÜR EINE REIHE THERMISCHER EINLAUFPROBLEME

Zusammenfassung—Es wird gezeigt, dass eine Reihe von thermischen Einlaufproblemen mit Hilfe der Laplace-Transformation als Variationsprobleme dargestellt werden können. Es handelt sich dabei um Probleme mit instationärer laminarer Strömung für konstante Stoffeigenschaften bei denenes möglich ist die Geschwindigkeitsprofile als voll ausgebildet anzusehen

Die Variationsdarstellung erlaubt auch die Anwendung des wirkungsvollen Ritz-Galerkin-Verfahrens, um Näherungslösungen zu erhalten. Das vorgeschlagene Verfahren wird durch die Anwendung auf das Graetz-Problem zwischen halbunendlichen parallelen Platten und auf das Eintrittsproblem bei sinusförmig sich ändernder Wandtemperatur veranschaulicht.

О ВАРИАЦИОННОЙ ФОРМУЛИРОВКЕ КЛАССА ЗАДАЧ ДЛЯ ТЕПЛОВОГО УЧАСТКА НА ВХОДЕ

Аннотация—Показано, что класс задач нестационарного ламинарного течения с постоянными свойствами на термическом входном участке, в которых профиль скорости можно считать полностью развитым, можно формулировать как вариационную задачу в области преобразования Лапласа. В свою очередь, вариационная формулировка позволяет применить мощную технику Ритца-Галеркина для получения приближенных решений. Предложенная техника иллюстрируется на примере задачи Гретца между полубесконечными параллельными пластинами и задачи о начальном участке, когда температура стенки меняется по синусоидальному закону.